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J. Math. Anal. Appl. 338 (2008) 1458–1468

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

A detailed analysis on local bifurcation from infinity for nonlinear elliptic problems

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Received 9 March 2007

Available online 20 June 2007

Submitted by Steven G. Krantz

Abstract

In this paper we analyze the local side of the bifurcation from infinity at the first eigenvalue of several elliptic operators. We underline that the key to decide the local behavior of the bifurcation lies in the sign of certain integral involving all the values of the nonlinearity.

Keywords: Bifurcation from infinity; Elliptic boundary value problems; Eigenvalues; Laplacian operator; p-Laplacian operator; Strong resonance

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary and consider the Dirichlet boundary value problem

$$\left.\begin{array}{ll} \mathcal{L}u(x) = f(\lambda, x, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{array}\right\} \tag{1}$$

where $\mathcal{L}$ is either a linear or a quasilinear elliptic operator. We are particularly interested in the usual Laplacian operator, the p-Laplacian operator and also in the operator $\mathcal{L}u = -\operatorname{div}(A(x,u)\nabla u)$.

We study the local behavior of continua of solutions bifurcating from infinity assuming abstract bifurcation techniques can be used. Specifically, we obtain conditions to ensure that the bifurcation starts to the left (respectively to the right) side of σ_1, the first eigenvalue of a suitable auxiliary problem (linearized for the Laplacian, homogeneous for the p-Laplacian).

The side of the bifurcation with the usual Laplacian operator,

$$\left.\begin{array}{ll} -\Delta u(x) = \lambda u(x) + g(x, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{array}\right\} \tag{2}$$

has been studied by several authors always assuming asymptotical conditions on the nonlinearity g. Thus, Ambrosetti and Hess [3] take a nonlinearity g which keeps away from zero at infinity. Later, Ambrosetti and Arcoya [1] and

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.06.019

Arcoya and Gámez [5] allow g to approach zero at infinity as an appropriate power. In [3] and [1] the autonomous case is considered, and for a suitable $\alpha \leqslant 1$, the sign of

$$\lim_{s\to\infty} g(s)s^{\alpha}$$

decides the side of the bifurcation. In [5], it is proved that such result is also valid for $\alpha \leqslant 2$, and false if $\alpha > 2$. They also allow g to depend on $x \in \Omega$. In such case the side of the bifurcation is decided, for some $\alpha < 2$, by the sign of the integral

$$\int_{\Omega} A_{\alpha}\phi_1^{1-\alpha}$$

where $A_{\alpha}(x) = \lim_{s\to\infty} g(x,s)s^{\alpha}$ and ϕ_1 is the first positive eigenfunction associated to σ_1.

Problem (2) has also been studied but with Neumann boundary conditions by the authors in [11]. In that paper, one can see the differences between both boundary conditions. In the Neumann case, in contrast with the Dirichlet case, the power α does not play any role and the side of the bifurcation is essentially decided by the sign of $\int_{\Omega} g(x,s)\,dx$ for s large enough (see [11, Theorem 2.1] for further details).

The one dimensional autonomous case with Dirichlet boundary conditions was treated in [12], where the authors pointed out that for the problem

$$\left.\begin{array}{ll} -u''(x) = \lambda u(x) + g(u(x)), & x \in (0,\pi), \\ u(0) = u(\pi) = 0 & \end{array}\right\} \tag{3}$$

the local behavior of the bifurcation is determined by the global shape of the nonlinearity g. Adapting the arguments used by Dancer (see [9]), we prove that the key to decide how the bifurcation behaves is the sign of

$$\int_0^{+\infty} g(s)s\,ds.$$

Such result covers all the mentioned asymptotical conditions above for the one dimensional case, in which the value of that integral becomes either ∞ or $-\infty$. Observe also that for $\alpha > 2$ the sign of such integral is not decided by the asymptotical behavior of $g(s)s^{\alpha}$.

In this paper, we generalize such result by considering the N-dimensional case and allowing g to depend on $x \in \Omega$. The side of the bifurcation will be decided by the sign of certain integral which depends, not only on the global shape of the nonlinearity g, but also on the geometry of the domain (see formula (5) and Theorem 1 below).

The quasilinear p-Laplacian operator is also submitted with similar techniques. Consider the problem

$$\left.\begin{array}{ll} -\Delta_p u(x) = \lambda |u(x)|^{p-2}u(x) + g(x,u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{array}\right\} \tag{4}$$

Among related papers, we can emphasize Ambrosetti, García Azorero and Peral [2] and also [1,5]. More recently, Drábek, Girg and Takác [10].

The third considered operator is $\mathcal{L}u = -\mathrm{div}(A(x,u)\nabla u)$. We follow ideas used by Arcoya, Carmona and Pellacci [4] who show how, though the operator is not homogeneous, bifurcation techniques can be applied. This kind of operators has also been studied with bifurcation techniques by Carmona and Suárez [8].

It is important to remark that *the faster g approaches zero at infinity, the more difficult is to determine the side of the bifurcation*. In a first step we will consider nonlinearities which are *very small* at infinity (see hypothesis (G) below). Later, we will use comparison arguments to cover a wider class of nonlinearities, including all in the previous cited papers (see Theorem 2 below).

In the study of each different operator we must add extra hypotheses which ensure that bifurcation techniques can be applied. So, in Section 2 the usual Laplacian operator is considered and the main result is proved. In Section 3, firstly the p-Laplacian and finally the $-\mathrm{div}(A(x,u)\nabla u)$ operator are treated to obtain similar fitting conclusions.

2. Results for the Laplacian operator

The purpose of this section is to determine the sign of $(\sigma_1 - \lambda)$ for (λ, u) a solution of (2) close to the bifurcation from $(\sigma_1, +\infty)$.

In order to ensure that bifurcation occurs, we assume the hypotheses:

$$\text{(B)}\quad \begin{cases} \bullet\ g:\overline{\Omega}\times\mathbb{R}\to\mathbb{R} \text{ is a Carathéodory function} \\ \quad \text{(i.e. continuous in } s\in\mathbb{R} \text{ for a.e. } x\in\Omega \text{ and measurable in } x\in\Omega,\ \forall s\in\mathbb{R}), \\ \bullet \text{ there exists } r>N \text{ and } C\in L^r(\Omega) \text{ such that } |g(x,s)|\leqslant C(x)(1+|s|),\ \text{for all } (x,s)\in\Omega\times\mathbb{R}, \\ \bullet\ \lim_{|s|\to\infty}\frac{g(x,s)}{s}=0 \text{ uniformly in } x\in\overline{\Omega}. \end{cases}$$

Under (B), every eigenvalue of the linearized problem, σ_k, with odd multiplicity is a bifurcation point from infinity (see [15]) and the weak solutions of (2) lie in the space $W^{2,r}(\Omega)$ continuously embedded in $C^1(\overline{\Omega})$ ($r>N$), which will be the natural space to work. We will denote by $\|\cdot\|$ the usual norm in $L^2(\Omega)$.

2.1. Small nonlinearities at infinity

Assume firstly that the nonlinearity g satisfies:

$$\text{(G)}\quad \begin{cases} \bullet\ \lim_{|s|\to+\infty} g(x,s)s^2=0 \text{ uniformly on } x\in\Omega, \\ \bullet \text{ there exists } f\in L^1([0,+\infty)) \text{ such that } |g(x,s)s|<f(s),\ \text{for } x \text{ in a neighborhood of } \partial\Omega, \\ \bullet\ g(x,s) \text{ is continuous in } x\in\partial\Omega. \end{cases}$$

It gives sense to the expression:

$$I=I(g):=\int_{\partial\Omega}\frac{1}{|\nabla\phi_1(y)|}\left(\int_0^{+\infty} g(y,s)s\,ds\right)dy, \tag{5}$$

where σ_1 and ϕ_1 are, respectively, the first eigenvalue and the first eigenfunction (with $\|\phi_1\|=1$) of the problem:

$$\left.\begin{array}{ll} -\Delta\phi=\lambda\phi, & x\in\Omega, \\ \phi(x)=0, & x\in\partial\Omega. \end{array}\right\}$$

We prove the following result.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2-boundary. Assume that hypotheses* (B) *and* (G) *hold. Let (λ_n,u_n) be a sequence of positive solutions of* (2) *bifurcating from (σ_1,∞). Then,*

$$\lim_{n\to\infty}(\sigma_1-\lambda_n)\|u_n\|^3=I.$$

As a consequence, if $I>0$, the bifurcation from $(\sigma_1,+\infty)$ is to the left of σ_1 and if $I<0$ the bifurcation from $(\sigma_1,+\infty)$ is to the right of σ_1.

Proof. In order to make clear the proof of this theorem, we introduce some notation and state two lemmas.

Let (λ,u) be a solution of (2). Multiplying by ϕ_1 and integrating by parts,

$$(\sigma_1-\lambda)\int_\Omega u(x)\phi_1(x)\,dx=\int_\Omega g\big(x,u(x)\big)\phi_1(x)\,dx. \tag{6}$$

It is known that ϕ_1 lies in the interior of the C^1-cone of positive functions and also, for any sequence of solutions of (2), (λ_n,u_n) bifurcating from $(\sigma_1,+\infty)$, one has $u_n/\|u_n\|\to\phi_1$ in $C^1(\overline{\Omega})$. Consequently, near the bifurcation point the solutions are strictly positive in Ω. Also,

$$\int_\Omega \frac{u_n}{\|u_n\|}\phi_1\to\int_\Omega\phi_1^2=1,$$

and then, by (6), the sign of $(\sigma_1-\lambda_n)$ can be calculated by taking limit in the following expression:

$$\lim_{n\to\infty}(\sigma_1-\lambda_n)\|u_n\|^3=\lim_{n\to\infty}(\sigma_1-\lambda_n)\|u_n\|^2\int_\Omega u_n\phi_1$$
$$=\lim_{n\to\infty}\|u_n\|^2\int_\Omega g(x,u_n)\phi_1. \tag{7}$$

We will follow the notation in [13, Appendix 14.6]. Using that $\Omega\subseteq\mathbb{R}^N$ is a bounded domain with C^2-boundary and taking ν_y the inner normal vector in $y\in\partial\Omega$, there exists $t_0>0$ such that

$$\varphi:\partial\Omega\times(0,t_0]\to\Omega,\qquad \varphi(y,t)=y+\nu_y t$$

is a difeomorphism into

$$\Gamma_0=\{x\in\Omega:\ \operatorname{dist}(x,\partial\Omega)\leqslant t_0\}, \tag{8}$$

and $|J_\varphi|$ (the Jacobian determinant of φ) satisfies $\lim_{t\to 0}|J_\varphi(y+\nu_y t)|=1$, uniformly in $y\in\partial\Omega$. In addition, using (G) such t_0 can be chosen such that $|g(x,s)s|<f(s)$, $\forall(x,s)\in\overline{\Gamma_0}\times\mathbb{R}$.

Lemma 1. *Consider a sequence $u_n\in C^1(\overline{\Omega})$ such that $\|u_n\|\to\infty$ and also $u_n/\|u_n\|\to\phi_1$ (C^1-convergence). Then*

(i) $$\lim_{n\to\infty}\left(\frac{\|u_n\|\phi_1(x)}{u_n(x)}\right)=1\quad\text{uniformly in } x\in\Gamma_0.$$

In other words,

$$\lim_{n\to\infty}\left(\frac{\|u_n\|\phi_1(y+\nu_y t)}{u_n(y+\nu_y t)}\right)=1\quad\text{uniformly in } (y,t)\in\partial\Omega\times(0,t_0].$$

(ii) $$\lim_{n\to\infty}\frac{\|u_n\|}{\frac{\partial u_n}{\partial\nu_y}(x)}=\frac{1}{\frac{\partial\phi_1}{\partial\nu_y}(x)}\quad\text{uniformly in } x=y+\nu_y t\in\Gamma_0.$$

Proof. (i) It follows from the uniform convergence $\frac{\nabla u_n}{\|u_n\|}\to\nabla\phi_1$ in $\overline{\Omega}$. In fact, for each $y\in\partial\Omega$, by using the "Cauchy mean value theorem," one has: $\forall n\geqslant 1$, $\forall t\in(0,t_0]$ there exists $c_{n,t,y}\in(0,t_0)$ such that

$$\frac{\|u_n\|\phi_1(y+\nu_y t)}{u_n(y+\nu_y t)}=\frac{\|u_n\|\frac{\partial}{\partial t}\phi_1(y+\nu_y c_{n,t,y})}{\frac{\partial}{\partial t}u_n(y+\nu_y c_{n,t,y})}=\frac{\|u_n\|\frac{\partial\phi_1}{\partial\nu_y}(y+\nu_y c_{n,t,y})}{\frac{\partial u_n}{\partial\nu_y}(y+\nu_y c_{n,t,y})},$$

which converges to 1 uniformly in $(y,c)\in\partial\Omega\times(0,t_0)$.

(ii) Use again the uniform convergence $\frac{\nabla u_n}{\|u_n\|}\to\nabla\phi_1$ in $\overline{\Omega}$. In particular, the inner product $\frac{\nabla u_n(x)}{\|u_n\|}\cdot\nu_y$ converges to $\nabla\phi_1(x)\cdot\nu_y$ for all $x=y+\nu_y t\in\Gamma_0$, i.e.

$$\frac{1}{\|u_n\|}\frac{\partial u_n}{\partial\nu_y}(y+\nu_y t)\to\frac{\partial\phi_1}{\partial\nu_y}(y+\nu_y t)\quad\text{uniformly in } y+\nu_y t\in\Gamma_0.$$

Taking into account Hopf's Lemma, $\nabla\phi_1(y)$ is uniformly away from zero in $\overline{\Gamma_0}$, and then

$$\frac{\|u_n\|}{\frac{\partial u_n}{\partial\nu_y}(y+\nu_y t)}\to\frac{1}{\frac{\partial\phi_1}{\partial\nu_y}(y+\nu_y t)}\quad\text{uniformly in } y+\nu_y t\in\Gamma_0.$$

and the proof of Lemma 1 is concluded. □

Lemma 2. *Assume that hypotheses* (B) *and* (G) *hold. Consider a sequence $u_n\in C^1(\overline{\Omega})$ such that $\|u_n\|\to\infty$ and $u_n/\|u_n\|\to\phi_1$ (C^1-convergence). Then,*

$$\lim_{n\to\infty}\|u_n\|^2\int_\Omega g(x,u_n)\phi_1=I.$$

Proof. We divide the domain in two parts, Γ_0 given in (8), and $\Omega \setminus \Gamma_0$.

$$\begin{aligned}\|u_n\|^2 \int_\Omega g\big(x, u_n(x)\big)\phi_1(x)\,dx &= \|u_n\|^2 \int_{\Omega\setminus\Gamma_0} g\big(x, u_n(x)\big)\phi_1(x)\,dx + \|u_n\|^2 \int_{\Gamma_0} g\big(x, u_n(x)\big)\phi_1(x)\,dx \\ &= I_1 + I_2.\end{aligned}$$

We claim that $I_1 \to 0$. Observe that since $\phi_1 \geqslant \varepsilon_0 > 0$ in $\Omega \setminus \Gamma_0$, then $u_n(x) \to \infty$ uniformly in $\Omega \setminus \Gamma_0$.

$$I_1 = \|u_n\|^2 \int_{\Omega\setminus\Gamma_0} g\big(x, u_n(x)\big)\phi_1(x)\,dx = \int_{\Omega\setminus\Gamma_0} g\big(x, u_n(x)\big)u_n^2 \frac{\|u_n\|^2\phi_1(x)}{u_n^2}\,dx,$$

with $\frac{\|u_n\|^2\phi_1}{u_n^2}$ uniformly converging to the bounded limit $1/\phi_1$. By hypothesis (G), $g(x, u_n)u_n^2 \to 0$ uniformly in $\Omega \setminus \Gamma_0$. Then $I_1 \to 0$.

To study I_2 we use the parametrization $x = \varphi(y, t) = y + \nu_y t$, for $x \in \Gamma_0$.

$$\begin{aligned}I_2 &= \|u_n\|^2 \int_{\Gamma_0} g\big(x, u_n(x)\big)\phi_1(x)\,dx \\ &= \int_{\partial\Omega} \int_0^{t_0} \|u_n\|^2 g\big(y + \nu_y t, u_n(y + \nu_y t)\big)\phi_1(y + \nu_y t)\big|J_\varphi(y + \nu_y t)\big|\,dt\,dy.\end{aligned}$$

Now for every $y \in \partial\Omega$ fixed, we make the one dimensional change of variables $t \mapsto s = u_n(y + \nu_y t)$, which is invertible with inverse $s \mapsto t_n(s)$ and with $dt = \frac{ds}{\frac{\partial u_n}{\partial \nu_y}(y+\nu_y t_n(s))}$. So,

$$\begin{aligned}&\int_0^{t_0} \|u_n\|^2 g\big(y + \nu_y t, u_n(y + \nu_y t)\big)\phi_1(y + \nu_y t)\big|J_\varphi(y + \nu_y t)\big|\,dt \\ &\quad = \int_0^{u_n(y+\nu_y t_0)} \left[g\big(y + \nu_y t_n(s), s\big)s \frac{\|u_n\|\phi_1(y + \nu_y t_n(s))}{u_n(y + \nu_y t_n(s))}\big|J_\varphi\big(y + \nu_y t_n(s)\big)\big| \frac{\|u_n\|}{\frac{\partial u_n}{\partial \nu_y}(y + \nu_y t_n(s))}\right] ds \\ &\quad = \int_0^{+\infty} \chi_{[0,u_n(y+\nu_y t_0)]}\left[g\big(y + \nu_y t_n(s), s\big)s \frac{\|u_n\|\phi_1(y + \nu_y t_n(s))}{u_n(y + \nu_y t_n(s))}\big|J_\varphi\big(y + \nu_y t_n(s)\big)\big| \frac{\|u_n\|}{\frac{\partial u_n}{\partial \nu_y}(y + \nu_y t_n(s))}\right] ds.\end{aligned}$$

So, for $y \in \partial\Omega$ and $s > 0$, then $s \in (0, u_n(y + \nu_y t_0))$ for n large enough, and

- $t_n(s) \in (0, t_0)$ and converges to 0 as n tends to infinity,
- using (G), $g(y + \nu_y t_n(s), s)\,s \to g(y, s)\,s$ with $|g(y + \nu_y t_n(s), s)\,s| < f(s)$,
- $|J_\varphi(y + \nu_y t_n(s))|$ is bounded and converges uniformly to 1,
- $\frac{\|u_n\|\phi_1(y+\nu_y t_n(s))}{u_n(y+\nu_y t_n(s))}$ is bounded and converges uniformly to 1 (see item (i) in previous lemma),
- $\frac{\|u_n\|}{\frac{\partial u_n}{\partial \nu_y}(y+\nu_y t_n(s))}$ converges uniformly to $\frac{1}{\frac{\partial\phi_1}{\partial\nu_y}(y)} = \frac{1}{|\nabla\phi_1(y)|}$ and consequently it is bounded (use item (ii) in previous lemma and the continuity of $\nabla\phi_1$ in $\partial\Omega$).

So there exists $K > 0$ such that for every $y \in \partial\Omega$ and $s > 0$,

$$\left|g\big(y + \nu_y t_n(s), s\big)s \frac{\|u_n\|\phi_1(y + \nu_y t_n(s))}{u_n(y + \nu_y t_n(s))}\big|J_\varphi\big(y + \nu_y t_n(s)\big)\big| \frac{\|u_n\|}{\frac{\partial u_n}{\partial \nu_y}(y + \nu_y t_n(s))}\right| < Kf(s).$$

We can apply the Lebesgue's bounded convergence theorem to obtain

$$I_2 \to \int_{\partial\Omega} \frac{1}{|\nabla\phi_1(y)|}\left(\int_0^{+\infty} g(y,s)s\,ds\right)dy = I$$

and the Lemma 2 is proved. □

We finish the proof of Theorem 1 simply by taking into account identity (7) and Lemma 2. □

2.2. Greater nonlinearities

We now use comparison arguments to cover a wider class of nonlinearities, including most of the cases previously cited. Instead of assuming (G), we will suppose the following hypothesis on g:

$$\text{(G2)}\quad \begin{cases} \text{There exists } \tilde g : \overline{\Omega}\times\mathbb{R}\to\mathbb{R} \text{ satisfying (B), (G), and} \\ \quad\text{either}\quad \text{(G2+) } I(\tilde g) > 0 \text{ and } g(x,s) \geqslant \tilde g(x,s),\ \forall(x,s)\in\overline{\Omega}\times\mathbb{R}, \\ \quad\text{or}\quad\ \ \text{(G2−) } I(\tilde g) < 0 \text{ and } g(x,s) \leqslant \tilde g(x,s),\ \forall(x,s)\in\overline{\Omega}\times\mathbb{R}, \end{cases}$$

where $I(\tilde g)$ is defined as in (5).

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2-boundary, assume that g satisfies* (B) *and* (G2). *Then, if* (G2+), *the bifurcation of solutions of problem* (2) *from* $(\sigma_1, +\infty)$ *is to the left of* σ_1 *and if* (G2−) *such bifurcation is to the right of* σ_1.

Proof. Consider (λ_n, u_n), a sequence of solutions of problem (2) bifurcating from $(\sigma_1, +\infty)$. Since $u_n/\|u_n\| \to \phi_1$ (C^1-convergence) and $\tilde g$ satisfies (G), we can apply Lemma 2 to obtain

$$\lim_{n\to\infty} \|u_n\|^2 \int_\Omega \tilde g(x,u_n)\phi_1 = I(\tilde g).$$

So, the sign of $\int_\Omega \tilde g(x,u_n)\phi_1$ is the same of $I(\tilde g)$ for n large enough. We now take into account formula (6). If (G2+), for n large enough,

$$(\sigma_1 - \lambda_n)\int_\Omega u_n\phi_1 = \int_\Omega g(x,u_n)\phi_1 \geqslant \int_\Omega \tilde g(x,u_n)\phi_1 > 0.$$

Conversely, if (G2−), for n large,

$$(\sigma_1 - \lambda_n)\int_\Omega u_n\phi_1 = \int_\Omega g(x,u_n)\phi_1 \leqslant \int_\Omega \tilde g(x,u_n)\phi_1 < 0.$$

Since $\int_\Omega u_n\phi_1 > 0$ for n large, the proof is concluded. □

Remarks.

1. Observe that Theorem 2 is a natural extension of known results in [1,3] and [5]. Even if $g(y,s)s \notin L^1([0,\infty))$, our hypotheses give sense to the expression $0 < \int_0^{+\infty} g(y,s)s\,ds \leqslant +\infty$ or $0 > \int_0^{+\infty} g(y,s)s\,ds \geqslant -\infty$.
   We also remark the case given in [5], where there exists $\alpha < 2$ such that $\lim_{s\to+\infty} g(x,s)s^\alpha = A_\alpha(x)$ and the sign of $\int_\Omega A_\alpha(x)\phi_1^{1-\alpha}$ decides the side of the bifurcation. We do not improve such result because it is not always possible to construct a function $\tilde g(x,s)$ less (respectively greater) than $g(x,s)$ in the conditions of (G2+) (respectively (G2−)).

2. All the arguments used in this section apply also if $g(x,+\infty)\perp\phi_1$, instead of $g(x,+\infty)=0$. To consider such cases, we simply add a new term to the nonlinearity and observe that formula (6) and the results in this section remain valid for the problem
$$\left.\begin{array}{ll} -\Delta u(x)=\lambda u(x)+h(x)+g(x,u(x)), & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega \end{array}\right\} \tag{9}$$
where $h\in L^2(\Omega)$ with $\int_\Omega h\phi_1=0$.
3. Previous results can be easily adapted for any sequence (λ_n,u_n) of solutions of (9) bifurcating from $(\sigma_1,-\infty)$, i.e. $u_n/\|u_n\|\to-\phi_1$. In fact, taking $k(x,s)=-g(x,-s)$, every pair $(\lambda_n,v_n)=(\lambda_n,-u_n)$ is a solution of
$$\left.\begin{array}{ll} -\Delta v(x)=\lambda v(x)-h(x)+k(x,v(x)), & x\in\Omega,\\ v(x)=0, & x\in\partial\Omega. \end{array}\right\} \tag{10}$$
Consequently, the side of the bifurcation will depend on the sign of
$$I^-:=I(k)=\int_{\partial\Omega}\frac{1}{|\nabla\phi_1(y)|}\left(\int_0^{+\infty}k(y,s)s\,ds\right)dy$$
$$=\int_{\partial\Omega}\frac{1}{|\nabla\phi_1(y)|}\left(\int_{-\infty}^{0}g(y,s)s\,ds\right)dy.$$

Taking into account Remark 3, we use the ideas in [5] to approach resonant problems in the first eigenvalue. If both bifurcations (from $+\infty$ and $-\infty$) at σ_1 start "to the same side," the problem
$$\left.\begin{array}{ll} -\Delta u(x)=\sigma_1 u(x)+h(x)+g(x,u(x)), & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega \end{array}\right\} \tag{11}$$
($h\in L^2(\Omega)$ with $\int_\Omega h\phi_1=0$) admits, at least, one solution (see [5, Theorem 19] for details).

Next corollary, referred to the problem (11), slightly improves [9, Theorems 1 and 2] and its N-dimensional statement [17, Theorem 2.2]. Under hypotheses (B) and (G) the problem belongs to the so-called strongly resonant problems (see Bartolo, Benci and Fortunato [7]). Furthermore, we consider an example where the results in [7] and [5] cannot be applied.

Corollary 1. *Assume the hypotheses* (B), (G) *and* $h\in L^2(\Omega)$ *with* $\int_\Omega h\phi_1=0$. *If* $\operatorname{sign}(I)=\operatorname{sign}(I^-)$, *the resonant problem* (11) *admits at least one solution.*

Example. Let us consider a particular nonlinearity in problem (11). Take the autonomous $g:\mathbb{R}\to\mathbb{R}$ defined by
$$g(s)=\frac{1-s}{1+s^4}.$$
Clearly, g satisfies hypotheses (B) and (G). Moreover,
$$I=\int_{\partial\Omega}\frac{dy}{|\nabla\phi_1(y)|}\int_0^{+\infty}g(s)s\,ds=\frac{1-\sqrt{2}}{4}\pi\int_{\partial\Omega}\frac{dy}{|\nabla\phi_1(y)|}<0,$$
$$I^-=\int_{\partial\Omega}\frac{dy}{|\nabla\phi_1(y)|}\int_{-\infty}^{0}g(s)s\,ds=\frac{-2-\sqrt{2}}{4\sqrt{2}}\pi\int_{\partial\Omega}\frac{dy}{|\nabla\phi_1(y)|}<0,$$
which implies, by using the corollary, that the problem (11) has, at least, one solution. However, we cannot apply neither [7] (because the integral value $\int_{-\infty}^{+\infty}g(s)\,ds=-\frac{\sqrt{2}}{2}\pi\neq0$) nor [5] (because g approaches zero at infinity quickly).

It is worth mentioning that the case of resonant problems in which the nonlinearity g is periodic with mean value zero can also be approached with bifurcation techniques, and convenient estimates on the bifurcation parameter, to obtain infinitely many solutions for convex bounded domains in $\mathbb{R}^N$ with $N\leqslant 2$ (see [16] and the references therein).

3. The extension to some quasilinear operators

3.1. The p-Laplacian operator

Consider now the quasilinear p-Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$. Specifically, consider the nonlinear boundary value problem,

$$\left.\begin{array}{ll} -\Delta_p u(x) = \lambda |u(x)|^{p-2}u(x) + g(x, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{array}\right\} \tag{12}$$

In order to ensure that bifurcation occurs we change hypothesis (B) by

$$(\mathrm{B}_p) \quad \begin{cases} \bullet\ g : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function,} \\ \bullet\ \text{there exists } r > N \text{ and } C \in L^r(\Omega) \text{ such that } |g(x,s)| \leqslant C(x)(1+|s|)^{p-1}, \text{ for all } (x,s) \in \Omega \times \mathbb{R}, \\ \bullet\ \lim_{|s|\to\infty} \frac{g(x,s)}{s^{p-1}} = 0 \text{ uniformly in } x \in \Omega. \end{cases}$$

Then (see [5]) $(\sigma_1, +\infty)$ is a bifurcation point in the sense that there exists a sequence of solutions of (12), (λ_n, u_n), with $\lambda_n \to \sigma_1$ and $\|u_n\|_{L^p} \to \infty$, $\frac{u_n}{\|u_n\|_{L^p}} \to \phi_1$, where σ_1 and ϕ_1 are, respectively, the first eigenvalue and the first eigenfunction of the problem

$$\left.\begin{array}{ll} -\Delta_p \phi = \lambda |\phi|^{p-2}\phi, & x \in \Omega, \\ \phi(x) = 0, & x \in \partial\Omega. \end{array}\right\}$$

In fact, σ_1 is characterized by

$$\frac{1}{\sigma_1} = \sup_{\phi \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega \phi^p}{\int_\Omega |\nabla\phi|^p}$$

and ϕ_1 lies in the interior of the cone of positive functions in $C^1(\overline{\Omega})$, with $\|\phi_1\|_{L^p} = 1$.

As in previous section, the purpose of this one is also to determine the sign of $(\sigma_1 - \lambda)$ for (λ, u) a solution of (12) close to $(\sigma_1, +\infty)$. In fact, the sign of such difference is also the sign of I defined by (5). The main results that we use to prove the next theorem are two estimates of $\sigma_1 - \lambda$ which play the role of formula (6) in this case.

By [5, Lemma 24], if (λ, u) is a solution of (12) with $u \in C^1(\overline{\Omega})$ and positive, then

$$\int_\Omega g(x,u)\frac{\phi_1^p}{u^{p-1}} \leqslant \sigma_1 - \lambda \leqslant \|u\|_{L^p}^{-p} \int_\Omega g(x,u)u. \tag{13}$$

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2-boundary, assume* (G), (B_p) *and let (λ_n, u_n) be a sequence of solutions of* (12) *bifurcating from $(\sigma_1, +\infty)$. Then,*

$$\lim \|u_n\|_{L^p}^{p+1}(\sigma_1 - \lambda_n) = I.$$

Proof. Use the same arguments that in previous section to estimate the outer terms in (13). □

3.2. The $-\operatorname{div}(A(x,u)\nabla u)$ operator

Consider the problem

$$\left.\begin{array}{ll} -\operatorname{div}(A(x,u)\nabla u(x)) = \lambda u(x) + g(x, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{array}\right\} \tag{14}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with C^2-boundary, $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies (B) and $A(x,s) := (a_{ij}(x,s))$, $i, j = 1, \dots, N$ is a symmetric matrix with Carathéodory coefficients $a_{ij} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ such that there exists positive constants α and β satisfying, for every $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$ and a.e. $x \in \Omega$,

$$(\mathrm{A1}) \quad \begin{cases} \bullet\ |A(x,s)| \leqslant \beta, \\ \bullet\ A(x,s)\xi \cdot \xi \geqslant \alpha|\xi|^2, \\ \bullet\ |A(x,s) - A(x,t)| \leqslant w(|s-t|),\ \forall s, t \in \mathbb{R} \end{cases}$$

where $A\xi \cdot \zeta$ means $\xi^T A\zeta$ and $w : \mathbb{R}^+ \to \mathbb{R}$ is an Osgood function, that is, w is not decreasing, $w(0) = 0$, $\int_{0+} \frac{ds}{w(s)} = +\infty$. Hypothesis (A1) ensures, for any $h \in H^{-1}(\Omega)$, the existence (by using items one an two, see [14]) and uniqueness (by third item, see [6]) of weak solution of the problem

$$\left.\begin{array}{ll} -\operatorname{div}(A(x,u)\nabla u(x)) = h, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{array}\right\}$$

i.e. $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega A(x,u)\nabla u \cdot \nabla v = \int_\Omega hv, \quad \forall v \in H_0^1(\Omega).$$

Note that it is suffices A to be Lipschitz in $s \in \mathbb{R}$ to satisfy the existence of an Osgood function, $w(s) = Ls$ (L the Lipschitz constant).

We define a solution of (14) as a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega A(x,u)\nabla u \cdot \nabla v = \lambda \int_\Omega uv + \int_\Omega g(x,u)v, \quad \forall v \in H_0^1(\Omega).$$

Under hypothesis

$$\text{(A2)} \quad \begin{cases} \bullet\ \exists \lim_{s\to+\infty} A(x,s) = A(x,+\infty) \text{ uniformly in } x \in \overline{\Omega}, \\ \bullet\ A \in C^1(\overline{\Omega} \times \mathbb{R}),\ A(\cdot,+\infty) \in C^1(\overline{\Omega}) \end{cases}$$

the eigenvalue problem

$$\left.\begin{array}{ll} -\operatorname{div}(A(x,+\infty)\nabla u(x)) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{array}\right\} \tag{15}$$

has a first positive eigenvalue σ_1 and a positive eigenfunction associated ϕ_1 with $\|\phi_1\| = 1$. In spite of the fact that this operator is not homogeneous, in [4] the authors overcome this difficulty and prove that $(\sigma_1, +\infty)$ is a bifurcation point of problem (14), in the sense that there exists a sequence (λ_n, u_n) such that $u_n/\|u_n\| \to \phi_1$ in $C^1(\overline{\Omega})$ and, consequently, near the bifurcation point, u_n are strictly positive.

One more time, the purpose is to determinate the sign of $(\sigma_1 - \lambda)$ when (λ, u) is a solution of (14) which is close to the bifurcation point. The main difference with previous cases is the participation in the expressions of the quadratic form $A(x,s)$. If (λ, u) is a solution of (14), taking $v = \phi_1$ and integrating by parts, we obtain,

$$(\sigma_1 - \lambda)\int_\Omega u\phi_1 = \int_\Omega \big[A(x,+\infty) - A(x,u)\big]\nabla u \cdot \nabla\phi_1 + \int_\Omega g(x,u)\phi_1. \tag{16}$$

Using the same arguments as in Section 2, the behavior of the bifurcation will be decided by the sign of the right-hand side of (16), which is divided in two terms. The second one has appeared in previous sections and does not need any extra treatment. The first one requires the definition of a new expression which has fundamental importance in the final result. Let us define J as

$$J := \int_{\partial\Omega} \big|\nabla\phi_1(y)\big| \left[\int_0^{+\infty} \big[A(y,+\infty) - A(y,s)\big]\,ds\right] \nu_y \cdot \nu_y\,dy. \tag{17}$$

The result in [4] shows that, assuming C^1-regularity on $A(x,s)$ and $A(x,+\infty)$, if the quadratic form induced by the matrix $A(x,+\infty) - A(x,s)$ is definite nonnegative and g is positive and "small enough" then the bifurcation is to the left of σ_1. Conversely, if the quadratic form is nonpositive and g is negative and "small enough" the bifurcation is to the right.

We improve such result by proving that the side of the bifurcation is decided by the sign of a joint expression calculated using both A and g, even if $A(x,+\infty) - A(x,s)$ is nondefinite and g is not "so small."

We will consider the next hypotheses which allow us to apply suitable convergence integral theorems.

$$\text{(A3)} \quad \begin{cases} \bullet\ \lim_{s\to+\infty} s\,[A(x,+\infty) - A(x,s)] = 0, \text{ uniformly in } x \in \overline{\Omega}, \\ \bullet\ \text{There exists } f \in L^1([0,+\infty)) \text{ such that} \\ \quad |a_{ij}(x,\infty) - a_{ij}(x,s)| < f(s),\ \forall x \in \Gamma_0,\ \forall s \in \mathbb{R}^+,\ \forall i,j = 1,\dots,N. \end{cases}$$

Lemma 3. *Assume* (A1), (A2), *and* (A3). *Let u_n be a sequence such that $u_n/\|u_n\| \to \phi_1$ in $C^1(\overline{\Omega})$, then*

$$\lim_{n\to+\infty} \int_\Omega \big[A(x,+\infty) - A(x,u_n)\big]\nabla u_n \cdot \nabla\phi_1 = J.$$

Proof. As in the proof of Lemma 2, we divide the domain in two parts, Γ_0 given in (8), and $\Omega \setminus \Gamma_0$.

$$\begin{aligned}
&\int_\Omega \big[A(x,+\infty) - A(x,u_n)\big]\nabla u_n \cdot \nabla\phi_1 = \\
&\quad = \int_{\Omega\setminus\Gamma_0} \big[A(x,+\infty) - A(x,u_n)\big]\nabla u_n \cdot \nabla\phi_1 + \int_{\Gamma_0} \big[A(x,+\infty) - A(x,u_n)\big]\nabla u_n \cdot \nabla\phi_1 \\
&\quad = I_1 + I_2.
\end{aligned}$$

We claim $I_1 \to 0$. Since, $u_n(x) > 0$, $\forall x \in \Omega \setminus \Gamma_0$,

$$\begin{aligned}
I_1 &= \int_{\Omega\setminus\Gamma_0} \big[A(x,+\infty) - A(x,u_n)\big]\nabla u_n \cdot \nabla\phi_1 \\
&= \int_{\Omega\setminus\Gamma_0} u_n\big[A(x,+\infty) - A(x,u_n)\big]\frac{\nabla u_n}{\|u_n\|} \cdot \nabla\phi_1 \frac{\|u_n\|}{u_n}
\end{aligned}$$

and, as before, by the pointwise convergence of the different terms and using (A3) we obtain $I_1 \to 0$.

Let us now prove that $I_2 = \int_{\Gamma_0}[A(x,+\infty) - A(x,u_n)]\nabla u_n \cdot \nabla\phi_1$ converges to J. Parameterizing Γ_0 by $\varphi(y,t) = y + \nu_y t$,

$$I_2 = \int_{\partial\Omega} \int_0^{t_0} \big[A(y+\nu_y t,+\infty) - A\big(y+\nu_y t, u_n(y+\nu_y t)\big)\big]\nabla u_n(y+\nu_y t) \cdot \nabla\phi_1(y+\nu_y t)\big|J_\varphi(y+\nu_y t)\big|\,dt\,dy,$$

and for any fixed $y \in \partial\Omega$, using the change of variables $t \mapsto s = u_n(y+\nu_y t)$ invertible with inverse $s \mapsto t_n(s)$ and $dt = \frac{1}{\frac{\partial u_n}{\partial\nu_y}(y+\nu_y t_n(s))}ds$

$$\begin{aligned}
I_2 = \int_{\partial\Omega} \int_0^{t_0} \Bigg[&\big[A\big(y+\nu_y t_n(s),+\infty\big) - A\big(y+\nu_y t_n(s), s\big)\big]\frac{\nabla u_n(y+\nu_y t_n(s))}{\|u_n\|} \\
&\cdot \nabla\phi_1\big(y+\nu_y t_n(s)\big)\big|J_\varphi\big(y+\nu_y t_n(s)\big)\big|\frac{\|u_n\|}{\frac{\partial u_n}{\partial\nu_y}(y+\nu_y t_n(s))}\Bigg]\,dt\,dy.
\end{aligned}$$

As in Theorem 1, hypotheses and Lemma 1 are applied to prove the convergence and the bound of the different functions. Therefore, the Lebesgue's bounded convergence theorem can be used to obtain,

$$I_2 \to \int_{\partial\Omega} \frac{1}{\frac{\partial\phi_1}{\partial\nu_y}(y)}\left[\int_0^{+\infty} \big[A(y,+\infty) - A(y,s)\big]\,ds\right]\nabla\phi_1(y) \cdot \nabla\phi_1(y)\,dy.$$

Since $\nabla\phi_1(y) = \frac{\partial\phi_1}{\partial\nu_y}(y)\nu_y$, we deduce, $I_2 \to J$. □

Theorem 4. *Assume* (A1), (A2), (A3), (B) *and also that there exists $G(x) \in L^1(\Omega)$ such that $|g(x,s)| < G(x)$, $\forall s \in [0,\infty)$, and there exists $g(x,+\infty) = \lim_{s\to+\infty} g(x,s)$, a.e. $x \in \Omega$. If (λ_n, u_n) is a sequence of solutions of* (14) *bifurcating from (σ_1,∞), then*

$$\lim_{n\to\infty} (\sigma_1 - \lambda_n)\|u_n\| = J + \int_\Omega g(x,+\infty)\phi_1,$$

where J is given by (17).

Proof. It is a consequence of previous lemma and results in previous section. □

Remarks.

1. Since $A(x,+\infty) - A(x,s)$ can change sign this result generalizes previous one in [4] where the quadratic form had to be definite.
2. Comparison methods allow to obtain similar results in the case

$$-\infty \leqslant J + \int_{\Omega} g(x,+\infty)\phi_1 \leqslant +\infty.$$

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